# Undecidable Properties on Length-Two String Rewriting Systems

Masahiko Sakai[1]

*Graduate School of Information Science, Nagoya University, furo-cho Chikusa-ku, Nagoya 4648603 Japan*

Yi Wang[2]

*Graduate School of Arts and Science, The University of Tokyo, Komaba Meguro-ku, Tokyo 1538914 Japan*

**Abstract**

Length-two string rewriting systems are length-preserving string rewriting systems that consist of length-two rules. This paper shows that confluence, termination, left-most termination and right-most termination are undecidable properties for length-two string rewriting systems. These results mean that these properties are undecidable for the class of linear term rewriting systems in which depth-two variables are allowed in both sides of rules.

## 1 Introduction

Confluence and termination are both generally undecidable for term rewriting systems (TRSs) and for string rewriting systems (SRSs). Hence several decidable classes have been studied. Confluence is a decidable property for terminating TRSs [12], and ground TRSs [16]. The latter result was extended to linear shallow TRSs [7] and shallow right-linear TRSs [8]. Classes for which termination is a decidable property are investigated and extended: ground TRSs [10], right-ground TRSs [4], TRSs that consist of right-ground rules, collapsing rules and shallow right-linear rules [9], and the related class of shallow left-linear TRSs [18].

Results on undecidable classes also exist. Confluence is an undecidable property for semi-constructor TRSs [14]. The result is extended to flat TRSs [11,15]. Termination is an undecidable property for three-rule SRSs [13], length-preserving SRSs [2] and one-rule TRSs [3].

[1] Email: sakai@is.nagoya-u.ac.jp

[2] Email: wangyi@graco.c.u-tokyo.ac.jp

SRSs are said to be length preserving if the left-hand side and the right-hand side of each rule have the same length. Since there is a finite number of rules, the number of different symbols appearing in the rules is finite, and fixed for SRSs, and the number of strings with a given length is also finite. Thus the decidability of the following problems for length-preserving SRSs trivially follows.

(i) *Reachability problem*: problem to decide $s \xrightarrow[\mathcal{R}]{*} t$ for given strings $s$ and $t$ and an SRS $\mathcal{R}$.

(ii) *String-confluence problem*: problem to decide confluence of $s$ for a given string $s$ and an SRS $\mathcal{R}$.

(iii) *String-termination problem*: problem to decide termination of $s$ for a given string $s$ and an SRS $\mathcal{R}$.

From these observation, one may think that confluence is also a decidable property; however, this is false. In this paper, we show that confluence, termination, left-most termination, and right-most termination are undecidable properties for length-two SRSs which are length-preserving SRSs consisting of length-two rules. First we show those properties for length-preserving SRSs by reducing Post's correspondence problem, which is known to be undecidable. Then we give a transformation of length-preserving SRSs to length-two SRSs that preserves those properties.

The class of length-two SRSs is a subclass of linear TRSs in which depth-two variables are allowed in both sides of the rules. Thus the undecidability for this class of TRSs is obtained. In that sense, the undecidability results in this paper shed new light on the borderline between decidability and undecidability for TRSs.

## 2 Preliminaries

Let $\Sigma$ be an alphabet. A *string rewrite rule* is a pair of strings $l$, $r \in \Sigma^*$, denoted by $l \to r$. A finite set of string rewrite rules is called a *string rewriting system* (SRS). A string is called a *redex* if it is the left-hand side of a rule. An SRS $\mathcal{R}$ induces a *rewrite step* relation $\xrightarrow[\mathcal{R}]{}$ defined as $s \xrightarrow[\mathcal{R}]{} t$ if there exist $u, v \in \Sigma^*$, and a rule $l \to r$ in $\mathcal{R}$ such that $s = ulv$ and $t = urv$. Especially the rewrite step is *left-most* (resp. *right-most*) if $l$ is the left-most (resp. right-most) redex in $s$. We use $\xleftrightarrow[\mathcal{R}]{}$ for $\xleftarrow[\mathcal{R}]{} \cup \xrightarrow[\mathcal{R}]{}$, $\xrightarrow[\mathcal{R}]{+}$ for the transitive closure of $\xrightarrow[\mathcal{R}]{}$ and $\xrightarrow[\mathcal{R}]{*}$ for the reflexive-transitive closure of $\xrightarrow[\mathcal{R}]{}$. We write $\xrightarrow[\mathcal{R}]{k}$ for the relation with $k$ rewrite steps. A (possibly infinite) sequence $s_0 \xrightarrow[\mathcal{R}]{} s_1 \xrightarrow[\mathcal{R}]{} \cdots$ is called a *reduction sequence (starting from $s_0$)*. We refer to $\{r \to l \mid l \to r \in \mathcal{R}\}$ by $\mathcal{R}^{-1}$.

We say that a string $s$ is *terminating* if every reduction sequence starting from $s$ is finite. We say that strings $s_1$ and $s_2$ are *joinable* if $s_1 \xrightarrow[\mathcal{R}]{*} s \xleftarrow[\mathcal{R}]{*} s_2$ for some $s$, denoted by $s_1 \downarrow_{\mathcal{R}} s_2$. A string $s$ is *confluent* if $s_1 \downarrow_{\mathcal{R}} s_2$ for any $s_1 \xleftarrow[\mathcal{R}]{*} s \xrightarrow[\mathcal{R}]{*} s_2$. An SRS $\mathcal{R}$ is *confluent* (resp. *terminating*) if all strings are confluent (resp. terminating).

In this paper, the notation $|u|$ represents the length of string $u$. The notation $a^m$ represents the string that consists of $m$ symbols of $a$. When we pay no heed to the number $m$, we denote $a^m$ as $a^*$ (if $m \geq 0$) or $a^+$ (if $m > 0$).

Now we recall Post's correspondence problem (PCP).

**Definition 2.1** *An instance of PCP is a finite set $P \subseteq \mathcal{A}^* \times \mathcal{A}^*$ of finite pairs of non-empty strings over an alphabet $\mathcal{A}$ with at least two symbols. A solution of $P$ is a string $w$ such that $w = u_1 \cdots u_k = v_1 \cdots v_k$ for some $(u_i, v_i) \in P$. The Post's correspondence problem (PCP) is the problem to decide whether such a solution exists or not.*

**Example 2.2** *The set $P = \{(ab, a), (c, bc)\}$ is an instance of PCP over $\{a, b, c\}$. It has a solution $abc = u_1u_2 = v_1v_2$ with $(u_1, v_1) = (ab, a)$, $(u_2, v_2) = (c, bc)$.*

**Theorem 2.3 ([17])** *PCP is undecidable.*

## 3 Length-preserving SRSs and undecidability of their termination

**Definition 3.1** *An SRS $\mathcal{R}$ is said to be length-preserving if $|l| = |r|$ for every rule $l \to r$ in $\mathcal{R}$.*

In this section we argue about the undecidability of termination, right-most termination and left-most termination for length-preserving SRSs. As stated in the introduction, Caron showed the undecidability in [2]. Moreover the proof works also for right-most termination and left-most termination because there is only one redex in each string that corresponds to a correct automata configuration. Nevertheless we give an alternative proof for the following reasons:

- Caron's proof is composed of two stages; the first stage gives an algorithm that reduces PCP into the uniform halting problem for linear-bounded automata and the second stage gives an algorithm reducing the uniform halting problem into the termination problem for length-preserving SRSs. On the other hand, we give a proof by reducing PCP into the termination problem of SRSs directly.
- The SRS $\mathcal{T}_P$ given in this section is rather straightforward and easy to understand. This helps the understanding of the SRS $\mathcal{C}_P$ given in the next section, which is more difficult although it is just a variant of $\mathcal{T}_P$.

As a preparation for giving the transformation, we introduce a kind of null symbol - and an equal length representation of each pair in instances of PCP. Let $P = \{(u_1, v_1), \ldots, (u_n, v_n)\}$ be an instance of PCP over $\mathcal{A}$.

$$\begin{aligned}\overline{P} &= \{(u, v\text{-}^m) \mid (u, v) \in P \text{ and } |u| - |v| = m \geq 0\} \\ &\quad \cup \{(u\text{-}^m, v) \mid (u, v) \in P \text{ and } |u| - |v| = -m < 0\}\end{aligned}$$

We write $\overline{\mathcal{A}}$ for $\mathcal{A} \cup \{\text{-}\}$. We define an equivalence relation $\sim \subseteq (\overline{\mathcal{A}})^* \times (\overline{\mathcal{A}})^*$ as the identity relation that ignores all null symbols -, that is $u \sim v$ if and only if $\hat{u} = \hat{v}$ where $\hat{u}$ and $\hat{v}$ denote the strings obtained from $u$ and $v$ by removing all -s respectively.

**Example 3.2** *For an instance $P = \{(ab, a), (c, bc)\}$ of PCP, we have $\overline{P} = \{(ab, a\text{-}), (c\text{-}, bc)\}$. The solution corresponds to $u_1u_2 = ab\ c\text{-} \sim a\text{-}\ bc = v_1v_2$ for $(u_1, v_1), (u_2, v_2) \in \overline{P}$.*

We use symbols like $0_{\substack{h\\a\\b\\a'\\b'}}$, where 0 is called the *state* of the symbol, $h$ is called the *first subscript* or *delimiter*, and $a$ is called the *second subscript* of the symbol, $a'$ the *third*, $b$ the *fourth*, and $b'$ the *fifth*. We code the solution of the previous example into $0_{\substack{h\\a\\a\\a\\a}}0_{\substack{i\\b\\-\\b\\-}}0_{\substack{h\\c\\b\\c\\b}}0_{\substack{i\\-\\c\\-\\c}}$, where each delimiter $h$ is used to represent that the corresponding second and third subscripts are head symbols of an element of the instance $P$.

For an easy handling of strings that consist of such symbols, we introduce a notation defined as $(X_1 \cdots X_k)_{\substack{h_1\cdots h_k\\a_1\cdots a_k\\b_1\cdots b_k\\a'_1\cdots a'_k\\b'_1\cdots b'_k}} = X_{1\substack{h_1\\a_1\\b_1\\a'_1\\b'_1}} \cdots X_{k\substack{h_k\\a_k\\b_k\\a'_k\\b'_k}}$. For example the above solution is denoted by $(00)_{\substack{hi\\ab\\a\text{-}\\ab\\a\text{-}}}(00)_{\substack{hi\\c\text{-}\\bc\\c\text{-}\\bc}}$ or $(0000)_{\substack{hihi\\abc\text{-}\\a\text{-}bc\\abc\text{-}\\a\text{-}bc}}$. Note that the lengths of the strings in those subscripts are the same whenever we use this notation. Hence we sometimes write $(0^+)_{\substack{hihi\\abc\text{-}\\a\text{-}bc\\abc\text{-}\\a\text{-}bc}}$ for the solution.

Delimiters and the second and third subscripts keep a candidate of solutions of $P$ in equal length representation and will never be changed by reductions. The fourth and fifth subscripts are used as a working area to check whether the candidate is a solution or not.

We relate a solution of the given instance of PCP with a loop in an infinite reduction sequence:

$$\Xi_0(0^+)_{\substack{hi^*\\u_1\\v_1\\u_1\\v_1}} \cdots (0^+)_{\substack{hi^*\\u_k\\v_k\\u_k\\v_k}} \Psi_0 \xrightarrow[\mathcal{T}_P]{*} \Xi_2(2^+)_{\substack{hi^*\\u_1\\v_1\\w_1\\w_1}} \cdots (2^+)_{\substack{hi^*\\u_k\\v_k\\w_k\\w_k}} \Psi_2 \xrightarrow[\mathcal{T}_P]{*} \Xi_0(0^+)_{\substack{hi^*\\u_1\\v_1\\u_1\\v_1}} \cdots (0^+)_{\substack{hi^*\\u_k\\v_k\\u_k\\v_k}} \Psi_0$$

(i) The former part checks whether $u_1 \cdots u_k \sim v_1 \cdots v_k$ by using the fourth and fifth subscripts as a working area.

(ii) The latter part checks whether $(u_1, v_1), \ldots, (u_n, v_n) \in \overline{P}$ and initializes the working area.

**Definition 3.3** *Let $P$ be an instance of PCP over $\mathcal{A}$. The SRS $\mathcal{T}_P$ over $\Sigma$ obtained from $P$ is defined as follows, where individual rules are shown in Figure 1.*

$$\Sigma = \{\Xi_i, \Psi_i \mid i \in \{0,1,2\}\} \cup \Sigma_c$$
$$\Sigma_c = \left\{ n_{\substack{d_1\\x_1\\x_2\\x_3\\x_4}}, \underline{n}_{\substack{d_1\\x_1\\x_2\\x_3\\x_4}}, \;\middle|\; d_1 \in \{h, i\},\, x_i \in \overline{\mathcal{A}},\, n \in \{0,1,2\} \right\}$$
$$\mathcal{T}_P = \alpha_1 \cup \beta_1 \cup \gamma_1 \cup \alpha_2 \cup \beta_2 \cup \gamma_2 \cup \delta_2$$

**Example 3.4** *Consider the instance $P = \{(ab, a), (c, bc)\}$ of PCP. Rules $\alpha_1$, $\beta_1$ depend on $P$ and the other rules depend only on the alphabet $\mathcal{A}$.*

$$\alpha_1 = \left\{ (\underline{1}2)_{\substack{h\ i\\a\ b\\a\ \text{-}\\x_1y_1\\x_2y_2}} \Psi_2 \to (00)_{\substack{hi\\ab\\a\text{-}\\ab\\a\text{-}}} \Psi_0,\ (\underline{1}2)_{\substack{h\ i\\c\ \text{-}\\b\ c\\x_1y_1\\x_2y_2}} \Psi_2 \to (00)_{\substack{hi\\c\text{-}\\bc\\c\text{-}\\bc}} \Psi_0 \;\middle|\; x_i, y_i \in \overline{\mathcal{A}} \right\}$$
$$\beta_1 = \left\{ (\underline{1}2)_{\substack{h\ i\\a\ b\\a\ \text{-}\\x_1y_1\\x_2y_2}} 2_{\substack{h\\z_1\\z_2\\z_3\\z_4}} \to (00)_{\substack{hi\\ab\\a\text{-}\\ab\\a\text{-}}} \underline{1}_{\substack{h\\z_1\\z_2\\z_3\\z_4}},\ (\underline{1}2)_{\substack{h\ i\\c\ \text{-}\\b\ c\\x_1y_1\\x_2y_2}} 2_{\substack{h\\z_1\\z_2\\z_3\\z_4}} \to (00)_{\substack{hi\\c\text{-}\\bc\\c\text{-}\\bc}} \underline{1}_{\substack{h\\z_1\\z_2\\z_3\\z_4}} \;\middle|\; x_i, y_i, z_i \in \overline{\mathcal{A}} \right\}$$

*$\mathcal{T}_P$ is not terminating since we can construct an infinite reduction sequence. We*

$$
\begin{aligned}
\alpha_1 &= \left\{ (\underline{1}2\cdots 2)_{\begin{smallmatrix} d\\ u\\ v\\ u'\\ v' \end{smallmatrix}} \Psi_2 \to (00\cdots 0)_{\begin{smallmatrix} d\\ u\\ v\\ u\\ v \end{smallmatrix}} \Psi_0 \;\middle|\; (u,v) \in \overline{P}, u', v' \in (\overline{\mathcal{A}})^*,\, d = hi^{|u|-1} \right\} \\
\beta_1 &= \left\{ (\underline{1}2\cdots 2)_{\begin{smallmatrix} d\\ u\\ v\\ u'\\ v' \end{smallmatrix}} 2_{\begin{smallmatrix} h\\ x_1\\ x_2\\ x_3\\ x_4 \end{smallmatrix}} \to (00\cdots 0)_{\begin{smallmatrix} d\\ u\\ v\\ u\\ v \end{smallmatrix}} \underline{1}_{\begin{smallmatrix} h\\ x_1\\ x_2\\ x_3\\ x_4 \end{smallmatrix}} \;\middle|\; (u,v) \in \overline{P}, u', v' \in (\overline{\mathcal{A}})^*, x_j \in \overline{\mathcal{A}},\, d = hi^{|u|-1} \right\} \\
\gamma_1 &= \left\{ \Xi_2 2_{\begin{smallmatrix} h\\ x_1\\ x_2\\ x_3\\ x_4 \end{smallmatrix}} \to \Xi_0 \underline{1}_{\begin{smallmatrix} h\\ x_1\\ x_2\\ x_3\\ x_4 \end{smallmatrix}} \;\middle|\; x_j \in \overline{\mathcal{A}} \right\} \\
\alpha_2 &= \left\{ 0_{\begin{smallmatrix} d_1\\ x_1\\ x_2\\ x_3\\ x_3 \end{smallmatrix}} \Psi_0 \to \underline{2}_{\begin{smallmatrix} d_1\\ x_1\\ x_2\\ x_3\\ x_3 \end{smallmatrix}} \Psi_2 \;\middle|\; d_1 \in \{h,i\},\, x_j \in \overline{\mathcal{A}} \right\} \\
\beta_2 &= \left\{ 0_{\begin{smallmatrix} d_1\\ x_1\\ x_2\\ x_3\\ x_3 \end{smallmatrix}} \underline{2}_{\begin{smallmatrix} d_2\\ y_1\\ y_2\\ y_3\\ y_3 \end{smallmatrix}} \to \underline{2}_{\begin{smallmatrix} d_1\\ x_1\\ x_2\\ x_3\\ x_3 \end{smallmatrix}} 2_{\begin{smallmatrix} d_2\\ y_1\\ y_2\\ y_3\\ y_3 \end{smallmatrix}}, \;\middle|\; d_j \in \{h,i\},\, x_j, y_j \in \overline{\mathcal{A}} \right\} \\
\gamma_2 &= \left\{ \Xi_0 \underline{2}_{\begin{smallmatrix} h\\ x_1\\ x_2\\ x_3\\ x_3 \end{smallmatrix}} \to \Xi_2 2_{\begin{smallmatrix} h\\ x_1\\ x_2\\ x_3\\ x_3 \end{smallmatrix}} \;\middle|\; x_j \in \overline{\mathcal{A}} \right\} \\
\delta_2 &= \left\{ 0_{\begin{smallmatrix} d_1\\ x_1\\ x_2\\ \text{-}\\ x_4 \end{smallmatrix}} 0_{\begin{smallmatrix} d_2\\ y_1\\ y_2\\ z\\ y_4 \end{smallmatrix}} \to 0_{\begin{smallmatrix} d_1\\ x_1\\ x_2\\ z\\ x_4 \end{smallmatrix}} 0_{\begin{smallmatrix} d_2\\ y_1\\ y_2\\ \text{-}\\ y_4 \end{smallmatrix}},\; 0_{\begin{smallmatrix} d_1\\ x_1\\ x_2\\ x_3\\ \text{-} \end{smallmatrix}} 0_{\begin{smallmatrix} d_2\\ y_1\\ y_2\\ y_3\\ z \end{smallmatrix}} \to 0_{\begin{smallmatrix} d_1\\ x_1\\ x_2\\ x_3\\ z \end{smallmatrix}} 0_{\begin{smallmatrix} d_2\\ y_1\\ y_2\\ y_3\\ \text{-} \end{smallmatrix}} \;\middle|\; d_j \in \{h,i\},\, x_j, y_j \in \overline{\mathcal{A}}, z \in \mathcal{A} \right\}
\end{aligned}
$$

Fig. 1. Rules in $\mathcal{T}_P$

*start with a string* $\Xi_0 (00)_{\begin{smallmatrix} hi\\ ab\\ a\text{-}\\ ab\\ a\text{-} \end{smallmatrix}} (00)_{\begin{smallmatrix} hi\\ c\text{-}\\ bc\\ c\text{-}\\ bc \end{smallmatrix}} \Psi_0$. *Rules in* $\delta_2$ *move null symbols in the fourth or fifth subscripts into the tail:*

$$
\Xi_0 (00)_{\begin{smallmatrix} hi\\ ab\\ a\text{-}\\ ab\\ a\text{-} \end{smallmatrix}} (00)_{\begin{smallmatrix} hi\\ c\text{-}\\ bc\\ c\text{-}\\ bc \end{smallmatrix}} \Psi_0 \underset{\delta_2}{\longrightarrow} \Xi_0 (00)_{\begin{smallmatrix} hi\\ ab\\ a\text{-}\\ ab\\ ab \end{smallmatrix}} (00)_{\begin{smallmatrix} hi\\ c\text{-}\\ bc\\ c\text{-}\\ \text{-}c \end{smallmatrix}} \Psi_0 \underset{\delta_2}{\longrightarrow} \Xi_0 (00)_{\begin{smallmatrix} hi\\ ab\\ a\text{-}\\ ab\\ ab \end{smallmatrix}} (00)_{\begin{smallmatrix} hi\\ c\text{-}\\ bc\\ c\text{-}\\ c\text{-} \end{smallmatrix}} \Psi_0.
$$

*Rules in* $\alpha_2 \cup \beta_2 \cup \gamma_2$ *check in right-to-left order that the fourth and fifth subscripts are the same:*

$$
\begin{aligned}
&\Xi_0 (00)_{\begin{smallmatrix} hi\\ ab\\ a\text{-}\\ ab\\ ab \end{smallmatrix}} (00)_{\begin{smallmatrix} hi\\ c\text{-}\\ bc\\ c\text{-}\\ c\text{-} \end{smallmatrix}} \Psi_0 \underset{\alpha_2}{\longrightarrow} \Xi_0 (00)_{\begin{smallmatrix} hi\\ ab\\ a\text{-}\\ ab\\ ab \end{smallmatrix}} (0\underline{2})_{\begin{smallmatrix} hi\\ c\text{-}\\ bc\\ c\text{-}\\ c\text{-} \end{smallmatrix}} \Psi_2 \underset{\beta_2}{\longrightarrow} \Xi_0 (00)_{\begin{smallmatrix} hi\\ ab\\ a\text{-}\\ ab\\ ab \end{smallmatrix}} (\underline{2}2)_{\begin{smallmatrix} hi\\ c\text{-}\\ bc\\ c\text{-}\\ c\text{-} \end{smallmatrix}} \Psi_2 \\
&\underset{\beta_2}{\longrightarrow} \Xi_0 (0\underline{2})_{\begin{smallmatrix} hi\\ ab\\ a\text{-}\\ ab\\ ab \end{smallmatrix}} (22)_{\begin{smallmatrix} hi\\ c\text{-}\\ bc\\ c\text{-}\\ c\text{-} \end{smallmatrix}} \Psi_2 \underset{\beta_2}{\longrightarrow} \Xi_0 (\underline{2}2)_{\begin{smallmatrix} hi\\ ab\\ a\text{-}\\ ab\\ ab \end{smallmatrix}} (22)_{\begin{smallmatrix} hi\\ c\text{-}\\ bc\\ c\text{-}\\ c\text{-} \end{smallmatrix}} \Psi_2 \underset{\gamma_2}{\longrightarrow} \Xi_2 (22)_{\begin{smallmatrix} hi\\ ab\\ a\text{-}\\ ab\\ ab \end{smallmatrix}} (22)_{\begin{smallmatrix} hi\\ c\text{-}\\ bc\\ c\text{-}\\ c\text{-} \end{smallmatrix}} \Psi_2.
\end{aligned}
$$

*Rules in* $\gamma_1 \cup \beta_1 \cup \alpha_1$ *check in left-to-right order that the second and third subscripts consist of pairs in* $\overline{P}$ *and copy the second subscript to the fourth and the third to the fifth respectively:*

$$
\Xi_2 (22)_{\begin{smallmatrix} hi\\ ab\\ a\text{-}\\ ab\\ ab \end{smallmatrix}} (22)_{\begin{smallmatrix} hi\\ c\text{-}\\ bc\\ c\text{-}\\ c\text{-} \end{smallmatrix}} \Psi_2 \underset{\gamma_1}{\longrightarrow} \Xi_0 (\underline{1}2)_{\begin{smallmatrix} hi\\ ab\\ a\text{-}\\ ab\\ ab \end{smallmatrix}} (22)_{\begin{smallmatrix} hi\\ c\text{-}\\ bc\\ c\text{-}\\ c\text{-} \end{smallmatrix}} \Psi_2 \underset{\beta_1}{\longrightarrow} \Xi_0 (00)_{\begin{smallmatrix} hi\\ ab\\ a\text{-}\\ ab\\ a\text{-} \end{smallmatrix}} (\underline{1}2)_{\begin{smallmatrix} hi\\ c\text{-}\\ bc\\ c\text{-}\\ c\text{-} \end{smallmatrix}} \Psi_2 \underset{\alpha_1}{\longrightarrow} \Xi_0 (00)_{\begin{smallmatrix} hi\\ ab\\ a\text{-}\\ ab\\ a\text{-} \end{smallmatrix}} (00)_{\begin{smallmatrix} hi\\ c\text{-}\\ bc\\ c\text{-}\\ bc \end{smallmatrix}} \Psi_0.
$$

Obviously $\mathcal{T}_P$ is length-preserving. The proof of the following lemma is found in Section 5.

**Lemma 3.5** *For an instance* $P$ *of PCP the following properties are equivalent:*

(i) $P$ *has a solution.*

(ii) $\mathcal{T}_P$ *is not right-most terminating.*

(iii) $\mathcal{T}_P$ *is not left-most terminating.*

(iv) $\mathcal{T}_P$ *is not terminating.*

**Theorem 3.6** *Termination, right-most termination and left-most termination are* *undecidable properties for length-preserving SRSs.*

**Proof.** We assume that termination (right-most termination, left-most termination) of length-preserving SRSs is decidable. Then it follows from Lemma 3.5 that PCP is decidable, which contradicts Theorem 2.3. □

## 4 Undecidability of confluence for length-preserving SRSs

We modify the construction of the SRS in the last section. In contrast to the SRS $\mathcal{T}_P$, which works sequentially, the SRS $\mathcal{C}_P$ works in parallel, that is, a solution of a given instance of PCP is related to the following two reduction sequences

$$\Xi_0(0^+)_{\substack{hi^* \\ u_1 \\ v_1 \\ u_1 \\ v_1}} \cdots (0^+)_{\substack{hi^* \\ u_k \\ v_k \\ u_k \\ v_k}} \Psi_0 \xrightarrow[\mathcal{C}_P]{*} \Xi_2(2^+)_{\substack{hi^* \\ u_1 \\ v_1 \\ w_1 \\ w_1}} \cdots (2^+)_{\substack{hi^* \\ u_k \\ v_k \\ w_k \\ w_k}} \Psi_2 \ ,$$

$$\Xi_0(0^+)_{\substack{hi^* \\ u_1 \\ v_1 \\ u_1 \\ v_1}} \cdots (0^+)_{\substack{hi^* \\ u_k \\ v_k \\ u_k \\ v_k}} \Psi_0 \xrightarrow[\mathcal{C}_P]{*} \Xi_1(1^+)_{\substack{hi^* \\ u_1 \\ v_1 \\ u_1 \\ v_1}} \cdots (1^+)_{\substack{hi^* \\ u_k \\ v_k \\ u_k \\ v_k}} \Psi_1$$

that demonstrate its non-confluence.

(i) The former reduction checks whether $u_1 \cdots u_k \sim v_1 \cdots v_k$ by using the fourth and fifth subscripts as a working area.

(ii) The latter reduction checks whether $(u_1, v_1), \ldots, (u_n, v_n) \in \overline{P}$, and checks that the working area is correctly initialized.

If $P$ has no solution then $\mathcal{C}_P$ must be confluent, which makes the design of $\mathcal{C}_P$ difficult.

**Definition 4.1** *Let $P$ be an instance of PCP over $\mathcal{A}$. The SRS $\mathcal{C}_P$ over $\Sigma$ obtained from $P$ is defined as follows:*

$$\mathcal{C}_P = \Theta \cup \Phi,$$

$$\Theta = \Theta_1 \cup \Theta_2, \quad \Phi = \gamma_1' \cup \gamma_2,$$

$$\Theta_1 = \alpha_1' \cup \beta_1' \cup (\alpha_1' \cup \beta_1')^{-1},$$

$$\Theta_2 = \alpha_2 \cup \beta_2 \cup \delta_2 \cup \epsilon_2 \cup (\alpha_2 \cup \beta_2 \cup \delta_2 \cup \epsilon_2)^{-1}$$

*where rules $\alpha_2$, $\beta_2$, $\delta_2$, and $\gamma_2$ are shown in Figure 1 and the other rules are shown in Figure 2.*

Remark that the reductions by $\Theta$-rules are symmetric, that is to say, $s \underset{\Theta}{\rightarrow} t$ if and only if $t \underset{\Theta}{\rightarrow} s$, which plays an important role in making $\mathcal{C}_P$ confluent when $P$ has no solution.

**Example 4.2** *Let $P = \{(ab, a), (c, bc)\}$ be an instance of PCP. Rules $\alpha_1'$, $\beta_1'$ depend*

$$
\begin{aligned}
\alpha'_1 &= \left\{ (00\cdots 0)\begin{smallmatrix}d\\u\\v\\u\\v\end{smallmatrix}\Psi_0 \to (\underline{1}1\cdots 1)\begin{smallmatrix}d\\u\\v\\u\\v\end{smallmatrix}\Psi_1 \;\middle|\; (u,v)\in\overline{P},\, d = hi^{|u|-1} \right\}\\
\beta'_1 &= \left\{ (00\cdots 0)\begin{smallmatrix}d\\u\\v\\u\\v\end{smallmatrix}\underline{1}\begin{smallmatrix}h\\x_1\\x_2\\x_1\\x_2\end{smallmatrix} \to (\underline{1}1\cdots 1)\begin{smallmatrix}d\\u\\v\\u\\v\end{smallmatrix}1\begin{smallmatrix}h\\x_1\\x_2\\x_1\\x_2\end{smallmatrix} \;\middle|\; (u,v)\in\overline{P}, x_i\in\overline{\mathcal{A}},\, d = hi^{|u|-1} \right\}\\
\gamma'_1 &= \left\{ \Xi_0\underline{1}\begin{smallmatrix}h\\x_1\\x_2\\x_1\\x_2\end{smallmatrix} \to \Xi_1 1\begin{smallmatrix}h\\x_1\\x_2\\x_1\\x_2\end{smallmatrix} \;\middle|\; x_i\in\overline{\mathcal{A}} \right\}\\
\epsilon_2 &= \left\{ 2\begin{smallmatrix}d_1\\x_1\\x_2\\\text{-}\\x_4\end{smallmatrix}2\begin{smallmatrix}d_2\\y_1\\y_2\\z\\y_4\end{smallmatrix} \to 2\begin{smallmatrix}d_1\\x_1\\x_2\\z\\x_4\end{smallmatrix}2\begin{smallmatrix}d_2\\y_1\\y_2\\\text{-}\\y_4\end{smallmatrix},\; 2\begin{smallmatrix}d_1\\x_1\\x_2\\x_3\\\text{-}\end{smallmatrix}2\begin{smallmatrix}d_2\\y_1\\y_2\\y_3\\z\end{smallmatrix} \to 2\begin{smallmatrix}d_1\\x_1\\x_2\\x_3\\z\end{smallmatrix}2\begin{smallmatrix}d_2\\y_1\\y_2\\y_3\\\text{-}\end{smallmatrix} \;\middle|\; d_j\in\{h,i\},\, x_j,y_j\in\overline{\mathcal{A}},\, z\in\mathcal{A} \right\}
\end{aligned}
$$

Fig. 2. Rules in $\mathcal{C}_P$

*on $P$ and the other rules depend only on the alphabet $\mathcal{A}$.*

$$
\begin{aligned}
\alpha'_1 &= \left\{ (00)\begin{smallmatrix}hi\\ab\\a\text{-}\\ab\\a\text{-}\end{smallmatrix}\Psi_0 \to (\underline{1}1)\begin{smallmatrix}hi\\ab\\a\text{-}\\ab\\a\text{-}\end{smallmatrix}\Psi_1,\; (00)\begin{smallmatrix}hi\\c\text{-}\\bc\\c\text{-}\\bc\end{smallmatrix}\Psi_0 \to (\underline{1}1)\begin{smallmatrix}hi\\c\text{-}\\bc\\c\text{-}\\bc\end{smallmatrix}\Psi_1 \right\}\\
\beta'_1 &= \left\{ (00)\begin{smallmatrix}hi\\ab\\a\text{-}\\ab\\a\text{-}\end{smallmatrix}\underline{1}\begin{smallmatrix}h\\x_1\\x_2\\x_1\\x_2\end{smallmatrix} \to (\underline{1}1)\begin{smallmatrix}hi\\ab\\a\text{-}\\ab\\a\text{-}\end{smallmatrix}1\begin{smallmatrix}h\\x_1\\x_2\\x_1\\x_2\end{smallmatrix},\; (00)\begin{smallmatrix}hi\\c\text{-}\\bc\\c\text{-}\\bc\end{smallmatrix}\underline{1}\begin{smallmatrix}h\\x_1\\x_2\\x_1\\x_2\end{smallmatrix} \to (\underline{1}1)\begin{smallmatrix}hi\\c\text{-}\\bc\\c\text{-}\\bc\end{smallmatrix}1\begin{smallmatrix}h\\x_1\\x_2\\x_1\\x_2\end{smallmatrix} \;\middle|\; x_i\in\overline{\mathcal{A}} \right\}
\end{aligned}
$$

*We can show that $\mathcal{C}_P$ is not confluent since we have non-joinable branches.*

$$
\begin{aligned}
&\Xi_0(00)\begin{smallmatrix}hi\\ab\\a\text{-}\\ab\\a\text{-}\end{smallmatrix}(00)\begin{smallmatrix}hi\\c\text{-}\\bc\\c\text{-}\\bc\end{smallmatrix}\Psi_0 \underset{\alpha'_1}{\longrightarrow} \Xi_0(00)\begin{smallmatrix}hi\\ab\\a\text{-}\\ab\\a\text{-}\end{smallmatrix}(\underline{1}1)\begin{smallmatrix}hi\\c\text{-}\\bc\\c\text{-}\\bc\end{smallmatrix}\Psi_1 \underset{\beta'_1}{\longrightarrow} \Xi_0(\underline{1}1)\begin{smallmatrix}hi\\ab\\a\text{-}\\ab\\a\text{-}\end{smallmatrix}(11)\begin{smallmatrix}hi\\c\text{-}\\bc\\c\text{-}\\bc\end{smallmatrix}\Psi_1\\
&\underset{\gamma'_1}{\longrightarrow} \Xi_1(11)\begin{smallmatrix}hi\\ab\\a\text{-}\\ab\\a\text{-}\end{smallmatrix}(11)\begin{smallmatrix}hi\\c\text{-}\\bc\\c\text{-}\\bc\end{smallmatrix}\Psi_1\;,\\
&\Xi_0(00)\begin{smallmatrix}hi\\ab\\a\text{-}\\ab\\a\text{-}\end{smallmatrix}(00)\begin{smallmatrix}hi\\c\text{-}\\bc\\c\text{-}\\bc\end{smallmatrix}\Psi_0 \;\overset{*}{\underset{\delta_2\cup\alpha_2\cup\beta_2\cup\gamma_2}{\longrightarrow}}\; \Xi_2(22)\begin{smallmatrix}hi\\ab\\a\text{-}\\ab\\ab\end{smallmatrix}(22)\begin{smallmatrix}hi\\c\text{-}\\bc\\c\text{-}\\c\text{-}\end{smallmatrix}\Psi_2.
\end{aligned}
$$

*Note that the detail of the latter sequence is found in Example 3.4.*

Obviously $\mathcal{C}_P$ is length preserving. The proof of the following main lemma is found in Section 5.

**Lemma 4.3** *Let $P$ be an instance of PCP. Then, $P$ has a solution if and only if $\mathcal{C}_P$ is not confluent.*

**Theorem 4.4** *Confluence of length-preserving SRSs is an undecidable property.*

**Proof.** We assume that the problem is decidable. Then it follows from Lemma 4.3 that PCP is decidable, which contradicts to Theorem 2.3. □

## 5 Proofs

Every occurrence of the symbols $\Xi_0$, $\Xi_1$, and $\Xi_2$ ($\Psi_0$, $\Psi_1$, and $\Psi_2$) in rules are left-most (right-most) positions in both sides. Moreover, for every rule, $\Xi_i$ ($\Psi_i$) appears in the left-hand side if and only if $\Xi_j$ ($\Psi_j$) appears in the right-hand side. Hence we can separate any reduction sequence having a symbol $\Xi_i$ ($\Psi_i$) into two reduction sequences by cutting each string at the left of $\Xi_i$ occurrence (at the right of $\Psi_i$ occurrence). Therefore the following proposition holds.

**Proposition 5.1** *Let $\mathcal{R}$ be $\mathcal{T}_P$ or $\mathcal{C}_P$ obtained from an instance $P$ of PCP. For any $i\in\{0,1,2\}$ and $S_1, S_2, S\in\Sigma^*$, the following hold:*

(a) *If* $S_1\Xi_i S_2 \underset{\mathcal{R}}{\rightarrow} S$*, then* $(S = S_1'\Xi_i S_2) \wedge (S_1 \underset{\mathcal{R}}{\rightarrow} S_1')$ *or* $(S = S_1\Xi_j S_2') \wedge (\Xi_i S_2 \underset{\mathcal{R}}{\rightarrow} \Xi_j S_2')$ *for some* $S_1', S_2' \in \Sigma^*$*, and* $j \in \{0, 1, 2\}$*.*

(b) *If* $S_1\Xi_i S_2 \overset{*}{\underset{\mathcal{R}}{\rightarrow}} S$*, then* $S = S_1'S_2'$*,* $S_1 \overset{*}{\underset{\mathcal{R}}{\rightarrow}} S_1'$*, and* $\Xi_i S_2 \overset{*}{\underset{\mathcal{R}}{\rightarrow}} S_2'$ *for some* $S_1' \in \Sigma^*$ *and non-empty* $S_2' \in \Sigma^*$*.*

(c) *If* $S_1\Psi_i S_2 \underset{\mathcal{R}}{\rightarrow} S$*, then* $(S = S_1'\Psi_j S_2) \wedge (S_1\Psi_i \underset{\mathcal{R}}{\rightarrow} S_1'\Psi_j)$ *or* $(S = S_1\Psi_i S_2') \wedge (S_2 \underset{\mathcal{R}}{\rightarrow} S_2')$ *for some* $S_1', S_2' \in \Sigma^*$ *and* $j \in \{0, 1, 2\}$*.*

(d) *If* $S_1\Psi_i S_2 \overset{*}{\underset{\mathcal{R}}{\rightarrow}} S$*, then* $S = S_1'S_2'$*,* $S_1\Psi_i \overset{*}{\underset{\mathcal{R}}{\rightarrow}} S_1'$*, and* $S_2 \overset{*}{\underset{\mathcal{R}}{\rightarrow}} S_2'$ *for some* $S_2' \in \Sigma^*$ *and non-empty* $S_1' \in \Sigma^*$*.*

**Proof.** We prove (a). Let $S_1\Xi_i S_2 \underset{\mathcal{R}}{\rightarrow} S$. The only interesting case is that the redex in the rewrite step contains the displayed symbol $\Xi_i$. Then one of $\gamma_1$-rules, $\gamma_2$-rules, or $\gamma_1'$-rules is applied. From the construction of the rules, we have $S = S_1\Xi_j S_2'$ and $\Xi_i S_2 \underset{\mathcal{R}}{\rightarrow} \Xi_j S_2'$ for some $S_2' \in \Sigma^*$ and $j \in \{0, 1, 2\}$.

The claim (b) is easily proved by induction on the number $k$ of the rewrite steps in $S_1\Xi_i S_2 \overset{*}{\underset{\mathcal{R}}{\rightarrow}} S$. For (c) and (d), the proofs are similar to (a) and (b) respectively.□

We say a string over $\Sigma$ is *normal* if it is in one of the following three forms:

(p1) $\Xi_i\chi$, (p2) $\chi\Psi_j$, (p3) $\Xi_i\chi\Psi_j$,

where $\chi \in (\Sigma_c)^*$, $i, j \in \{0, 1, 2\}$.

We prepare a measure for the proof of the next lemma. For a non-empty string $X_1 \cdots X_n$ over $\Sigma$, we define $||X_1 \cdots X_n||$ by the summation of the number of occurrences of $\Xi_i$ symbols in $X_2 \cdots X_n$, and the number of occurrences of $\Psi_i$ symbols in $X_1 \cdots X_{n-1}$.

**Lemma 5.2** *Let* $\mathcal{R}$ *be* $\mathcal{T}_P$ *or* $\mathcal{C}_P$ *over* $\Sigma$ *obtained from an instance* $P$ *of PCP. Then* $\mathcal{R}$ *is confluent (resp. terminating, right-most terminating, left-most terminating) if and only if* $w$ *is confluent (resp. terminating, right-most terminating, left-most terminating) for every normal* $w \in \Sigma^*$*.*

**Proof.** First we prove the termination part of the lemma. Since ⇒-direction is trivial, consider ⇐-direction.

Let $S_1 \underset{\mathcal{R}}{\rightarrow} S_2 \underset{\mathcal{R}}{\rightarrow} \cdots$ be an infinite reduction sequence starting from a non-normal string $S_1$ such that $||S_1||$ is minimal. We show a contradiction. We have two cases in which $S_1 = w\Xi_i S'$ and $S_1 = S'\Psi_i w$ for some normal $w$ and some $S' \in \Sigma^*$.

- In the former case, where $S_1 = w\Xi_i S'$, we can construct an infinite reduction sequence starting from at least one of $w$ or $\Xi_i S'$ by applying Proposition 5.1(a) infinitely many times, which contradicts the minimality of $S_1$.
- In the latter case, we can show a contradiction similar to the former case by using Proposition 5.1(c).

Secondly we prove the confluence part of the lemma. Since ⇒-direction is trivial, consider ⇐-direction. We show that every $S_1 \in \Sigma^+$ is confluent by induction on $||S_1||$. If $||S_1|| = 0$, then $S_1$ is normal and it is confluent from the assumption. If $||S_1|| > 0$, then we have two cases, in which $S_1 = w_1\Xi_i S_1'$ and $S_1 = S_1'\Psi_i w_1$ for some normal $w_1$ and some $S'_1 \in \Sigma^*$.

- In the former case, let $S_2 \overset{*}{\underset{\mathcal{R}}{\leftarrow}} w_1 \Xi_i S'_1 \overset{*}{\underset{\mathcal{R}}{\rightarrow}} S_3$. By Proposition 5.1(b), we have $S_2 = w_2 S'_2$, $S_3 = w_3 S'_3$, $w_2 \overset{*}{\underset{\mathcal{R}}{\leftarrow}} w_1 \overset{*}{\underset{\mathcal{R}}{\rightarrow}} w_3$ and $S'_2 \overset{*}{\underset{\mathcal{R}}{\leftarrow}} \Xi_i S'_1 \overset{*}{\underset{\mathcal{R}}{\rightarrow}} S'_3$. Since $w_1$ is confluent from the assumption, we have $w_2 \downarrow_{\mathcal{R}} w_3$. Since $\Xi_i S'_1$ is confluent from the induction hypothesis, we have $S'_2 \downarrow_{\mathcal{R}} S'_3$. Therefore we have $S_2 = w_2 S'_2 \downarrow_{\mathcal{R}} w_3 S'_3 = S_3$.
- In the latter case, we can show the confluence of $S_1$ by using Proposition 5.1(d) in a similar way to the former case.

□

Note that this lemma is provable more elegantly by using a notion of persistency [19] similarly to [5,6]. However we proved it without the notion to make the paper self-contained.

Thanks to Lemma 5.2, we can concentrate on normal strings in the rest of this section.

### *5.1 Termination analysis of $\mathcal{T}_P$*

In the sequel, we analyze the termination property for $\mathcal{T}_P$. We use the notation $\vec{u}$ for $u_1 \cdots u_k$ and $\vec{H}$ for $hi^{|u_1|-1} \cdots hi^{|u_k|-1}$.

**Lemma 5.3** *Let $P$ be an instance of PCP.*

*(a) If $u_1 \cdots u_k \sim v_1 \cdots v_k$ for some $(u_i, v_i) \in \overline{P}$, then $w \overset{+}{\underset{\mathcal{T}_P}{\rightarrow}} w$ where $w = \Xi_0 (0^+)^{hi^*}_{\substack{u_1\\v_1\\u_1\\v_1}} \cdots (0^+)^{hi^*}_{\substack{u_k\\v_k\\u_k\\v_k}} \Psi_0$. Moreover, both right-most reduction and left-most reduction are possible.*

*(b) If $\Xi_0 \chi \Psi_0 \overset{+}{\underset{\mathcal{T}_P}{\rightarrow}} \Xi_0 \chi \Psi_0$ for some $\chi \in (\Sigma_c)^*$, then $P$ has a solution.*

**Proof.** (a): We have a left-most reduction sequence $\Xi_0 (0^+)^{\vec{H}}_{\substack{\vec{u}\\\vec{v}\\\vec{u}\\\vec{v}}} \Psi_0 \overset{*}{\underset{\delta_2}{\rightarrow}} \Xi_0 (0^+)^{\vec{H}}_{\substack{\vec{u}\\\vec{v}\\\vec{w}\\\vec{w}}} \Psi_0 \overset{+}{\underset{\alpha_2 \cup \beta_2 \cup \gamma_2}{\rightarrow}} \Xi_2 (2^+)^{\vec{H}}_{\substack{\vec{u}\\\vec{v}\\\vec{w}\\\vec{w}}} \Psi_2$. Here the right-most reduction also exists by applying rules $\delta_2$ as lazily as possible. Since $(u_i, v_i) \in \overline{P}$, we have a left-most and right-most reduction sequence $\Xi_2 (2^+)^{\vec{H}}_{\substack{\vec{u}\\\vec{v}\\\vec{w}\\\vec{w}}} \Psi_2 \overset{+}{\underset{\gamma_1 \cup \beta_1 \cup \alpha_1}{\rightarrow}} \Xi_0 (0^+)^{\vec{H}}_{\substack{\vec{u}\\\vec{v}\\\vec{u}\\\vec{v}}} \Psi_0$.

(b): Let $\Xi_0 \chi \Psi_0 \overset{+}{\underset{\mathcal{T}_P}{\rightarrow}} \Xi_0 \chi \Psi_0$. From the construction of $\mathcal{T}_P$, a string $\Xi_2 \chi' \Psi_2$ must appear in this reduction sequence. From the reduction sequence $\Xi_0 \chi \Psi_0 \overset{+}{\underset{\mathcal{T}_P}{\rightarrow}} \Xi_2 \chi' \Psi_2$, the string $\chi$ is of the form $(0^+)^{hi^*}_{\substack{u_1\\v_1\\u'_1\\v'_1}} \cdots (0^+)^{hi^*}_{\substack{u_k\\v_k\\u'_k\\v'_k}}$ or $\chi$ contains $\underline{2}$. In the latter case, the reduction sequence $\Xi_2 \chi' \Psi_2 \overset{+}{\underset{\mathcal{T}_P}{\rightarrow}} \Xi_0 \chi \Psi_0$ is impossible. Thus, $\chi$ is of the form displayed above. From the reduction sequence $\Xi_0\chi\Psi_0 = \Xi_0 (0^+)^{\substack{hi^*\\u_1\\v_1}}_{\substack{u'_1\\v'_1}} \cdots (0^+)^{\substack{hi^*\\u_k\\v_k}}_{\substack{u'_k\\v'_k}} \Psi_0 \xrightarrow[\mathcal{T}_P]{+} \Xi_2\chi'\Psi_2$, $\chi'$ must be of the form $(2^+)^{\substack{hi^*\\u_1\\v_1}}_{\substack{w_1\\w_1}} \cdots (2^+)^{\substack{hi^*\\u_k\\v_k}}_{\substack{w_k\\w_k}}$ and $\vec{u'} \sim \vec{v'}$. From the reduction sequence $\Xi_2\chi'\Psi_2 = \Xi_2 (2^+)^{\substack{\vec{H}\\\vec{u}\\\vec{v}}}_{\substack{\vec{w}\\\vec{w}}} \Psi_2 \xrightarrow[\gamma_1\cup\beta_1\cup\alpha_1]{+} \Xi_0 (0^+)^{\substack{\vec{H}\\\vec{u}\\\vec{v}}}_{\substack{\vec{u'}\\\vec{v'}}} \Psi_0 = \Xi_0\chi\Psi_0$, we have $(u_i, v_i) \in \overline{P}$ for every $i$. Since $\vec{u'}$ and $\vec{v'}$ are copied from $\vec{u}$ and $\vec{v}$ respectively in the latter reduction sequence by $\beta_1$-rules, we have $\vec{u'} = \vec{u}$ and $\vec{v'} = \vec{v}$. Thus we conclude $\vec{u} \sim \vec{v}$, which means that $P$ has a solution. □

**Proof for Lemma 3.5**

((i)⇒(ii)∧(iii)): By Lemma 5.3(a).
((ii)∨(iii)⇒(iv)): Trivial.
((iv)⇒(i)): Let $\mathcal{T}_P$ not be terminating. From Lemma 5.2, there is a non-terminating and normal string $w$. Infinite reduction sequences starting from $w$ must contain a string starting with $\Xi_0$ and ending with $\Psi_0$ by the construction of $\mathcal{T}_P$. Thus the lemma follows from Lemma 5.3(b). □

*5.2 Confluence analysis of $\mathcal{C}_P$*

In the sequel, we analyze the confluence property for $\mathcal{C}_P$. The following propositions on the working area are obtained from the construction of rules.

**Proposition 5.4** *If* $(\cdots)^{\substack{d\\u\\v}}_{\substack{u'\\v'}} \xrightarrow[\mathcal{C}_P]{*} (\cdots)^{\substack{d\\u\\v}}_{\substack{u''\\v''}}$, *then* $u' \sim u''$ *and* $v' \sim u''$.

**Proposition 5.5** $\xleftarrow[\Theta]{*} \;=\; \xleftrightarrow[\Theta]{*} \;=\; \xrightarrow[\Theta]{*}$ .

The following lemma shows that strings in a specific form are closed under reductions by Θ-rules.

**Lemma 5.6** *Let* $m, n \geq 0$ *and* $p \in \{1, 2\}$. *If* $\chi = (0^n \underline{p}\, p^m)^{\substack{d\\u\\v}}_{\substack{u'\\v'}} \xrightarrow[\Theta]{*} \chi'$ *then* $\chi' = (0^{n'} \underline{p}\, p^{m'})^{\substack{d\\u\\v}}_{\substack{u''\\v''}}$ *for some* $m', n' \geq 0$ *and* $u'', v''$.

**Proof.** For any string in forms of $\chi$ for $p = 1$ (resp. $p = 2$), only $\Theta_1$-rules (resp. $\Theta_2$-rules) are applicable, which produce a string in forms of $\chi'$. □

We state some properties on $\Theta_1$-rules.

**Lemma 5.7** *Consider the following strings for* $i \leq j$*:*

$$\chi = (0^+)^{\substack{hi^*\\u_1\\v_1}}_{\substack{u'_1\\v'_1}} \cdots (0^+)^{\substack{hi^*\\u_{i-1}\\v_{i-1}}}_{\substack{u'_{i-1}\\v'_{i-1}}} (\underline{1}1^*)^{\substack{hi^*\\u_i\\v_i}}_{\substack{u'_j\\v'_i}} (1^+)^{\substack{hi^*\\u_{i+1}\\v_{i+1}}}_{\substack{u'_{i+1}\\v'_{i+1}}} \cdots (1^+)^{\substack{hi^*\\u_k\\v_k}}_{\substack{u'_k\\v'_k}},$$

$$\chi' = (0^+)^{\substack{hi^*\\u_1\\v_1}}_{\substack{u''_1\\v''_1}} \cdots (0^+)^{\substack{hi^*\\u_{j-1}\\v_{j-1}}}_{\substack{u''_{j-1}\\v''_{j-1}}} (\underline{1}1^*)^{\substack{hi^*\\u_j\\v_j}}_{\substack{u''_j\\v''_j}} (1^+)^{\substack{hi^*\\u_{j+1}\\v_{j+1}}}_{\substack{u''_{j+1}\\v''_{j+1}}} \cdots (1^+)^{\substack{hi^*\\u_k\\v_k}}_{\substack{u''_k\\v''_k}}.$$

*If* $\chi \xrightarrow[\Theta]{*} \chi'$ *then* $u_l = u'_l$, $v_l = v'_l$, *and* $(u_l, v_l) \in \overline{P}$ *for all* $i \leq l < j$ *and* $u'_l = u''_l$ *and* $v'_l = v''_l$ *for all* $j \leq l$.

**Proof.** The lemma is proved by induction on the number of the rewrite steps. $\square$

Next we state some properties on $\Theta_2$-rules.

**Lemma 5.8** *Let* $\chi = (\underline{2}2^*)^{\substack{d\\u\\v}}_{\substack{u'\\v'}} \xrightarrow[\Theta]{*} (0^*\underline{2})^{\substack{d\\u\\v}}_{\substack{u''\\v''}} = \chi'$. *Then* $u'' \sim u' \sim v' \sim v''$.

**Proof.** We can prove, by induction on $n$, the claim that $\chi \xrightarrow[\Theta]{n} (0^*\underline{2})^{\substack{d'\\u_1\\v_1}}_{\substack{u'_1\\v'_1}} (2^*)^{\substack{d''\\u_2\\v_2}}_{\substack{u'_2\\v'_2}}$ implies $u'_1 \sim v'_1$. Hence the lemma follows from Proposition 5.4. $\square$

**Lemma 5.9** *If* $w = \Xi_0 (0^+)^{\substack{hd\\u\\v}}_{\substack{u'\\v'}} \Psi_0 \xrightarrow[\mathcal{C}_P]{*} \Xi_0 \underline{2}^{\substack{h\\x_1\\x_2}}_{\substack{x_3\\x_3}} \chi \Psi_2 = w'$ *for some* $\chi \in (\Sigma_c)^*$, *then* $u' \sim v'$.

**Proof.** We prove the lemma by induction on the number of rewrite steps in the reduction sequence. In the case in which the first step is a reduction by $\alpha'_1$-rules, we have $w \xrightarrow[\alpha'_1]{} \Xi_0 \chi' \Psi_1 \xrightarrow[\Theta_1]{*} \Xi_0 \chi'' \Psi_1 \xrightarrow[(\alpha'_1)^{-1}]{} \Xi_0 (0^+)^{\substack{hd\\u\\v}}_{\substack{u''\\v''}} \Psi_0 \xrightarrow[\mathcal{C}_P]{*} w'$. The claim follows since $u' \sim u''$ and $v' \sim v''$ by Proposition 5.4 and $u'' \sim v''$ by the induction hypothesis.

Consider the case in which the first step is a reduction by $\alpha_2$-rules. We have $w \xrightarrow[\alpha_2]{} \Xi_0 (0^*\underline{2})^{\substack{hd\\u\\v}}_{\substack{u'\\v'}} \Psi_2 \xrightarrow[\mathcal{C}_P]{*} w'$. If $(\alpha_2)^{-1}$-rules are applied in the sequence then we can show the claim in a similar way to the case in which the first step is a reduction by $\alpha'_1$-rules. Hence assume that $(\alpha_2)^{-1}$-rules are not applied. Then, $w' = \Xi_0 (\underline{2}2^*)^{\substack{hd\\u\\v}}_{\substack{u''\\v''}} \Psi_2$ by Lemma 5.6. Thus $u' \sim v'$ follows from Proposition 5.5 and Lemma 5.8.

Consider the case in which the first step is a reduction by $\delta_2$-rules. We have $w \xrightarrow[\delta_2]{} \Xi_0 (0^+)^{\substack{hd\\u\\v}}_{\substack{u''\\v''}} \Psi_0 \xrightarrow[\mathcal{C}_P]{*} w'$. The claim follows since $u' \sim u''$ and $v' \sim v''$ from Proposition 5.4 and $u'' \sim v''$ from the induction hypothesis. $\square$

**Lemma 5.10** *If* $w = \Xi_0 (0^+)^{\substack{hi^*\\u_1\\v_1}}_{\substack{u'_1\\v'_1}} \cdots (0^+)^{\substack{hi^*\\u_k\\v_k}}_{\substack{u'_k\\v'_k}} \Psi_0 \xrightarrow[\mathcal{C}_P]{*} \Xi_0 \underline{1}^{\substack{h\\x_1\\x_2}}_{\substack{x_1\\x_2}} \chi \Psi_1 = w'$ *for some* $\chi \in (\Sigma_c)^*$, *then* $u_1 \cdots u_k \sim u'_1 \cdots u'_k$, $v_1 \cdots v_k \sim v'_1 \cdots v'_k$ *and* $(u_i, v_i) \in \overline{P}$ *for every* $i$.

**Proof.** We prove the lemma by induction on the number of rewrite steps in the reduction sequence. We only consider the case in which the first step is a reduction by $\alpha'_1$-rules and $(\alpha'_1)^{-1}$-rules are not applied in the sequence since the other cases that the first step is a reduction by $\alpha_2$-rules, or the first step is a reduction by $\alpha'_1$-rules and $(\alpha'_1)^{-1}$-rules are applied in the sequence are proved in a similar way to Lemma 5.9 by using Proposition 5.4 and the induction hypothesis.

We have $w \underset{\alpha'_1}{\rightarrow} w'' \overset{*}{\underset{\mathcal{C}_P}{\rightarrow}} w'$, $u_k = u'_k$ and $v_k = v'_k$, where $w'' = \Xi_0 \overset{hi^* \cdots hi^*}{(0^+)}{}^{\substack{u_1 \cdots u_{k-1} \\ v_1 \cdots v_{k-1}}}_{\substack{u'_1 \cdots u'_{k-1} \\ v'_1 \cdots v'_{k-1}}} \overset{hi^*}{(\underline{1}1^*)}{}^{\substack{u_k \\ v_k}}_{\substack{u'_k \\ v'_k}} \Psi_1$. Hence $w' = \Xi_0 \overset{hi^*}{(\underline{1}1^*)}{}^{\substack{u_1 \\ v_1}}_{\substack{u''_1 \\ v''_1}} \overset{hi^* \cdots hi^*}{(1^+)}{}^{\substack{u_2 \cdots u_k \\ v_2 \cdots v_k}}_{\substack{u''_2 \cdots u''_k \\ v''_2 \cdots v''_k}} \Psi_1$ by Lemma 5.6. By applying Lemma 5.7 with $i = 0$ and $j = k$ we obtain $u_l = u''_l$ and $v_l = v''_l$ for all $1 \leq l < k$ and $u''_k = u'_k$ and $v''_k = v'_k$. Hence we have $\vec{u} = \vec{u''}$ and $\vec{v} = \vec{v''}$. Since $\vec{u'} \sim \vec{u''}$ and $\vec{v'} \sim \vec{v''}$ by Proposition 5.4, $\vec{u} \sim \vec{u'}$ and $\vec{v} \sim \vec{v'}$ follow. □

**Lemma 5.11** *Let $P$ be an instance of PCP. If $w = \Xi_0 \overset{h}{\underline{1}}{}^{\substack{x_1 \\ x_2}}_{\substack{x_1 \\ x_2}} \chi \Psi_1 \overset{*}{\underset{\mathcal{C}_P}{\leftrightarrow}} \Xi_0 \overset{h}{\underline{2}}{}^{\substack{x_1 \\ x_2}}_{\substack{x_3 \\ x_3}} \chi' \Psi_2 = w'$ for some $\chi, \chi' \in (\Sigma_c)^*$, then $P$ has a solution.*

**Proof.**

Let $w \overset{*}{\underset{\mathcal{C}_P}{\leftrightarrow}} w'$. Then a string in forms of $\Xi_0 \chi'' \Psi_0$ must appear in this reduction and no underlined state appears in $\chi''$ from the construction of rules. Thus $\chi''$ must be in forms of $\Xi_0 \overset{hi^*}{(0^+)}{}^{\substack{u_1 \\ v_1}}_{\substack{u_1 \\ v_1}} \cdots \overset{hi^*}{(0^+)}{}^{\substack{u_k \\ v_k}}_{\substack{u_k \\ v_k}} \Psi_0$; otherwise the underlined state $\underline{1}$ displayed in $w$ does not move to the next symbol of $\Psi_i$ by Lemma 5.6 and the construction of rules. By Lemma 5.9 and Lemma 5.10, we have $\vec{u} \sim \vec{v}$ and $(u_i, v_i) \in \overline{P}$, which means $P$ has a solution. □

We need more lemmas to guarantee the confluence of $\mathcal{C}_P$ when $P$ has no solution.

**Lemma 5.12** *Let $w_1$ and $w_2$ be normal strings over $\Sigma^*$. Then,*

*(a) $w_1 \overset{*}{\underset{\mathcal{C}_P \setminus \gamma'_1}{\leftrightarrow}} w_2$ implies $w_1 \downarrow_{\mathcal{C}_P} w_2$, and*

*(b) $w_1 \overset{*}{\underset{\mathcal{C}_P \setminus \gamma_2}{\leftrightarrow}} w_2$ implies $w_1 \downarrow_{\mathcal{C}_P} w_2$.*

**Proof.** Before proving (a), we show the claim $(*)$ that $w_1 \underset{\gamma_2}{\leftarrow} w_2 \overset{*}{\underset{\Theta}{\rightarrow}} w_3 \underset{\gamma_2}{\rightarrow} w_4$ implies $w_1 \overset{*}{\underset{\Theta}{\rightarrow}} w_4$ by induction on the number of rewrite steps in $w_2 \overset{*}{\underset{\Theta}{\rightarrow}} w_3$. Here $w_2$ must begin with $\Xi_0 \overset{h}{\underline{2}}{}^{\substack{x_1 \\ x_2}}_{\substack{x_3 \\ x_3}}$ since it has a redex of $\gamma_2$. Hence we can represent that $w_1 = \Xi_2 \overset{hd}{(2^+)}{}^{\substack{u \\ v}}_{\substack{u' \\ v'}} S'$, $w_2 = \Xi_0 \overset{hd}{(\underline{2}2^*)}{}^{\substack{u \\ v}}_{\substack{u' \\ v'}} S'$, $w_3 = \Xi_0 \overset{hd}{(\underline{2}2^*)}{}^{\substack{u \\ v}}_{\substack{u'' \\ v''}} S''$ and $w_4 = \Xi_2 \overset{hd}{(2^+)}{}^{\substack{u \\ v}}_{\substack{u'' \\ v''}} S''$ for $S', S'' \in \Sigma^*$, where $n \neq 2$ for the left-most symbol $\overset{d}{n}{}^{\substack{x_1 \\ x_2}}_{\substack{x_3 \\ x_4}}$ of $S'$ and $S''$.

In the case where $S' = S'' = \Psi_2$, we have $w_1 \overset{*}{\underset{\epsilon_2 \cup \epsilon_2^{-1}}{\rightarrow}} w_4$ since $u' \sim u''$ and $v' \sim v''$ by Proposition 5.4. In the other cases, we can separate the reduction, from the construction of rules, into $S' \overset{*}{\underset{\Theta}{\rightarrow}} S''$ and $w'_1 = \Xi_2 \overset{hd}{(2^+)}{}^{\substack{u \\ v}}_{\substack{u' \\ v'}} \underset{\gamma_2}{\leftarrow} \Xi_0 \overset{hd}{(\underline{2}2^*)}{}^{\substack{u \\ v}}_{\substack{u' \\ v'}} \overset{*}{\underset{\Theta}{\rightarrow}} \Xi_0 \overset{hd}{(\underline{2}2^*)}{}^{\substack{u \\ v}}_{\substack{u'' \\ v''}} \underset{\gamma_2}{\rightarrow} \Xi_2 \overset{hd}{(2^+)}{}^{\substack{u \\ v}}_{\substack{u'' \\ v''}} = w'_4$. For the latter sequence, we have $w'_1 \overset{*}{\underset{\epsilon_2 \cup \epsilon_2^{-1}}{\rightarrow}} w'_4$ since $u' \sim u''$ and $v' \sim v''$ by Proposition 5.4. Therefore $w_1 \overset{*}{\underset{\Theta}{\rightarrow}} w_4$.

Now we prove (a) by induction on the number $k$ of reduction steps by $\gamma_2$-rules in $w_1 \overset{*}{\underset{\mathcal{C}_P \setminus \gamma_1'}{\leftrightarrow}} w_2$.

- ($k = 0$): We have $w_1 \overset{*}{\underset{\Theta}{\rightarrow}} w_2$ by Proposition 5.5.
- ($k = 1$): The reduction sequence can be represented as $w_1 \overset{*}{\underset{\Theta}{\leftrightarrow}} w_3 \underset{\gamma_2}{\leftrightarrow} w_4 \overset{*}{\underset{\Theta}{\leftrightarrow}} w_2$. Then $w_1 \overset{*}{\underset{\Theta}{\rightarrow}} w_3$ and $w_4 \overset{*}{\underset{\Theta}{\leftarrow}} w_2$ follow from Proposition 5.5.
- ($k > 1$): The reduction sequence can be represented as $w_1 \overset{*}{\underset{\Theta}{\leftrightarrow}} w_3 \underset{\gamma_2}{\leftrightarrow} w_4 \overset{*}{\underset{\mathcal{C}_P \setminus \gamma_1'}{\leftrightarrow}} w_2$. If $w_3 \underset{\gamma_2}{\rightarrow} w_4$ then $w_1 \downarrow_{\mathcal{C}_P} w_2$ follows from Proposition 5.5 and the induction hypothesis. Otherwise $w_1 \overset{*}{\underset{\Theta}{\leftrightarrow}} w_3 \underset{\gamma_2}{\leftarrow} w_4 \overset{*}{\underset{\Theta}{\leftrightarrow}} w_4' \underset{\gamma_2}{\rightarrow} w_2' \overset{*}{\underset{\mathcal{C}_P \setminus \gamma_1'}{\leftrightarrow}} w_2$. Since we have $w_1 \overset{*}{\underset{\Theta}{\leftrightarrow}} w_3 \overset{*}{\underset{\Theta}{\leftrightarrow}} w_2' \overset{*}{\underset{\mathcal{C}_P \setminus \gamma_1'}{\leftrightarrow}} w_2$ by claim ($*$) above, $w_1 \downarrow_{\mathcal{C}_P} w_2$ follows from the induction hypothesis and Proposition 5.5

The lemma (b) can be shown in a similar way to (a) by using the following claim ($**$). We show the claim ($**$) in which $w_1 \underset{\gamma_1'}{\leftarrow} w_2 \overset{*}{\underset{\Theta}{\rightarrow}} w_3 \underset{\gamma_1'}{\rightarrow} w_4$ implies $w_1 \overset{*}{\underset{\Theta}{\rightarrow}} w_4$ by induction on the number of rewrite steps in $w_2 \overset{*}{\underset{\Theta}{\rightarrow}} w_3$. Here $w_2$ must begin with $\Xi_0(\underline{1})^{\substack{h\\x_1\\x_2}}_{\substack{x_1\\x_2}}$ since it has a redex of $\gamma_1'$. Hence we can write $w_1 = \Xi_1(1^+)^{\substack{hd\\u\\v}}_{\substack{u'\\v'}} S'$, $w_2 = \Xi_0(\underline{1}1^*)^{\substack{hd\\u\\v}}_{\substack{u'\\v'}} S'$, $w_3 = \Xi_0(\underline{1}1^*)^{\substack{hd\\u\\v}}_{\substack{u''\\v''}} S''$ and $w_4 = \Xi_1(1^+)^{\substack{hd\\u\\v}}_{\substack{u''\\v''}} S''$ for $S', S'' \in \Sigma^*$, where $n \neq 1$ for the left-most symbol $n^{\substack{d\\x_1\\x_2}}_{\substack{x_3\\x_4}}$ of $S'$ and $S''$.

In the case where $S' = S'' = \Psi_1$, we have $u' = u''$ and $v' = v''$ by applying Lemma 5.7 with $i = j = 1$. Thus $w_1 = w_4$ follows. In the other cases, we can separate the reduction, from the construction of rules, into $S' \overset{*}{\underset{\Theta}{\rightarrow}} S''$ and $w_1' = \Xi_1(1^+)^{\substack{hd\\u\\v}}_{\substack{u'\\v'}} \underset{\gamma_1'}{\leftarrow} \Xi_0(\underline{1}1^*)^{\substack{hd\\u\\v}}_{\substack{u'\\v'}} \overset{*}{\underset{\Theta}{\rightarrow}} \Xi_0(\underline{1}1^*)^{\substack{hd\\u\\v}}_{\substack{u''\\v''}} \underset{\gamma_1'}{\rightarrow} \Xi_1(1^+)^{\substack{hd\\u\\v}}_{\substack{u''\\v''}} = w_4'$. For the latter sequence, we have $w_1' = w_4'$ since $u' \sim u''$ and $v' \sim v''$ by Lemma 5.7. Therefore $w_1 \overset{*}{\underset{\Theta}{\rightarrow}} w_4$. □

### Proof for Lemma 4.3

Since the $\Rightarrow$-direction is easy from the observation of Example 4.2, we show the $\Leftarrow$-direction. Assuming that $P$ has no solution, we show that $\mathcal{C}_P$ is confluent. From Lemma 5.2, considering $w_1 \overset{*}{\underset{\mathcal{C}_P}{\leftarrow}} w_0 \overset{*}{\underset{\mathcal{C}_P}{\rightarrow}} w_2$ for a normal string $w_0$ is enough.

- Consider the case in which $w_0$ starts with $\Xi_0$ and ends with $\Psi_i$ for some $i \in \{0, 1, 2\}$. Assume that both $\gamma_1'$ and $\gamma_2$ are applied in the reduction sequence. Then $P$ must have a solution by Lemma 5.11, which is a contradiction. Hence at least one of $\gamma_1'$ or $\gamma_2$ rules cannot be applied in the reduction sequence.
- In either of following cases:
  - · $w_0$ ends with $\Psi_i$ for some $i \in \{0, 1, 2\}$, and all other symbols are of $\Sigma_c$,

· $w_0$ starts with $\Xi_1$ or $\Xi_2$, and
· $w_0$ starts with $\Xi_0$ and all other symbols are of $\Sigma_c$.
It is easy to see that at least one of $\gamma_1'$ or $\gamma_2$ rules cannot be applied in the reduction sequence.

In any of the above cases, we have $w_1 \downarrow_{R_P} w_2$ by Lemma 5.12. □

# 6 Length-two SRSs

*Length-two SRSs* are SRSs that consist of rules with length two, that is, $|l| = |r| = 2$ for every rule $l \to r$. In this section we give a transformation of a length-preserving SRS over $\Sigma_0$ into a length-two SRS over $\Delta$ that preserves the confluence property and termination property.

Let $\Sigma = \Sigma_0 \cup \{\text{-}\}$ and $m + 1 (\geq 3)$ be the maximum length of rules in $\mathcal{R}$. Let $\Delta_0 = (\Sigma_0)^m$ and $\Delta = \Delta_0 \cup \{wv \mid w \in (\Sigma_0)^k, v = \text{-}^{m-k}, 1 \leq k \leq m-1\}$.

The natural mapping $\phi : \Delta \to \Sigma^m$ is defined as $\phi(w) = w$. This mapping is naturally extended to $\phi : \Delta^* \to \Sigma^*$.

**Example 6.1** *Let* $\Sigma_0 = \{a, b\}$ *and* $m = 2$. *Then* $\Delta_0 = \{aa, ab, ba, bb\}$, $\Delta = \Delta_0 \cup \{a\text{-}, b\text{-}\}$ *and* $\phi(ab\ bb\ a\text{-}) = abbba\text{-}$.

We give a transformation of a length-preserving SRS $\mathcal{R}$ into a length-two SRS $tw(\mathcal{R})$ over $\Delta$.

$$tw(\mathcal{R}) = \{w_1 w_2 \to w_3 w_4 \mid w_i \in \Delta,\ \phi(w_1 w_2) \underset{\mathcal{R}}{\to} \phi(w_3 w_4)\}.$$

**Example 6.2** *Let* $\mathcal{R} = \{bbb \to aaa\}$ *over* $\Sigma_0 = \{a, b\}$. *Then* $tw(\mathcal{R})$ *is the following length-two SRS over* $\Delta$, *where* $\Delta$ *is displayed in Example 6.1.*

$$tw(\mathcal{R}) = \begin{cases} bb\ b\text{-} \to aa\ a\text{-}, & bb\ ba \to aa\ aa, \quad bb\ bb \to aa\ ab, \\ ab\ bb \to aa\ aa, & bb\ bb \to ba\ aa \end{cases}$$

We say a string $w_1 \cdots w_n$ over $\Delta^*$ is *normal* if $w_1, \ldots, w_{n-1} \in \Delta_0$. From the construction of $tw(\mathcal{R})$, all reachable strings from a normal string are also normal.

We define a mapping $\psi : \Delta^* \to (\Sigma_0)^*$ as $\psi(\alpha) = w$ where $w$ is a string obtained from $\phi(\alpha)$ by removing all -'s. We define a mapping $\psi^{-1} : (\Sigma_0)^* \to \Delta^*$ as $\psi^{-1}(w) = \alpha$ where $\psi(\alpha) = w$ and $\alpha$ is normal. For example $\psi(ab\ bb\ a\text{-}) = abbba$ and $\psi^{-1}(abbba) = ab\ bb\ a\text{-}$. Trivially we have $\psi^{-1}(\psi(\alpha)) = \alpha$ for normal $\alpha \in \Delta^*$ and $\psi(\psi^{-1}(w)) = w$ for $w \in (\Sigma_0)^*$.

**Proposition 6.3** *(a) For a normal* $\alpha_1 \in \Delta^*$, *if* $\alpha_1 \underset{tw(\mathcal{R})}{\to} \alpha_2$ *then* $\psi(\alpha_1) \underset{\mathcal{R}}{\to} \psi(\alpha_2)$.

*(b) For* $w_1 \in (\Sigma_0)^*$, *if* $w_1 \underset{\mathcal{R}}{\to} w_2$ *then* $\psi^{-1}(w_1) \underset{tw(\mathcal{R})}{\to} \psi^{-1}(w_2)$.

**Proof.** From the construction of $tw(\mathcal{R})$. □

**Lemma 6.4** *For an SRS* $\mathcal{R}$, *the SRS* $tw(\mathcal{R})$ *is confluent (resp. terminating, right-most terminating, left-most terminating) if and only if* $\alpha$ *is confluent (resp. terminating, right-most terminating, left-most terminating) for every normal* $\alpha \in \Delta^*$.

**Proof.** We can prove the lemma in a similar way to the proof of Lemma 5.2. Here

$\Delta\backslash\Delta_0$ symbols play the same role as $\Psi_i$ symbols. Actually, every occurrence of the symbols in $\Delta\backslash\Delta_0$ in rules are right-most positions in both sides. Moreover, for every rule, a symbol in $\Delta\backslash\Delta_0$ appears in the left-hand side if and only if it appears in the right-hand side. Hence we can separate any reduction sequence having a symbol in $\Delta\backslash\Delta_0$ into several reduction sequences by cutting each string at the right of its occurrence without any effect for the properties. □

**Lemma 6.5** *Let $\mathcal{R}$ be an length-preserving SRS. $\mathcal{R}$ is terminating (resp. left-most terminating, right-most terminating) if and only if $tw(\mathcal{R})$ is terminating (resp. left-most terminating, right-most terminating).*

**Proof.** (⇒): Let $tw(\mathcal{R})$ be non-terminating. By Lemma 6.4 we have an infinite reduction sequence for $tw(\mathcal{R})$ starting from a normal string. This direction follows from Proposition 6.3(a).
(⇐): Let $\mathcal{R}$ be non-terminating. Then we have an infinite reduction sequence. By Proposition 6.3(b) we have an infinite reduction sequence for $tw(\mathcal{R})$.

This proof also works on either left-most cases or right-most cases. □

**Lemma 6.6** *Let $\mathcal{R}$ be a length-preserving SRS. $\mathcal{R}$ is confluent if and only if $tw(\mathcal{R})$ is confluent.*

**Proof.** (⇒): Let $\beta_1 \overset{*}{\underset{tw(\mathcal{R})}{\leftarrow}} \alpha \overset{*}{\underset{tw(\mathcal{R})}{\rightarrow}} \beta_2$. We can assume that $\alpha$ is normal by Lemma 6.4. We have $\psi(\beta_1) \overset{*}{\underset{\mathcal{R}}{\leftarrow}} \psi(\alpha) \overset{*}{\underset{\mathcal{R}}{\rightarrow}} \psi(\beta_2)$ by Proposition 6.3(a). Since $\mathcal{R}$ is confluent, there exists a string $w \in \Sigma_0^*$ such that $\psi(\beta_1) \overset{*}{\underset{\mathcal{R}}{\rightarrow}} w \overset{*}{\underset{\mathcal{R}}{\leftarrow}} \psi(\beta_2)$. Therefore we have $\beta_1 = \psi^{-1}(\psi(\beta_1)) \overset{*}{\underset{tw(\mathcal{R})}{\rightarrow}} \psi^{-1}(w) \overset{*}{\underset{tw(\mathcal{R})}{\leftarrow}} \psi^{-1}(\psi(\beta_2)) = \beta_2$ by Proposition 6.3(b).
(⇐): Let $u_1 \overset{*}{\underset{\mathcal{R}}{\leftarrow}} w \overset{*}{\underset{\mathcal{R}}{\rightarrow}} u_2$. We have $\psi^{-1}(u_1) \overset{*}{\underset{tw(\mathcal{R})}{\leftarrow}} \psi^{-1}(w) \overset{*}{\underset{tw(\mathcal{R})}{\rightarrow}} \psi^{-1}(u_2)$ by Proposition 6.3(b). Since $tw(\mathcal{R})$ is confluent, there exists a string $\alpha \in \Delta^*$ such that $\psi^{-1}(u_1) \overset{*}{\underset{tw(\mathcal{R})}{\rightarrow}} \alpha \overset{*}{\underset{tw(\mathcal{R})}{\leftarrow}} \psi^{-1}(u_2)$. Since $\alpha$ is normal, we have $u_1 = \psi(\psi^{-1}(u_1)) \overset{*}{\underset{\mathcal{R}}{\rightarrow}} \psi(\alpha) \overset{*}{\underset{\mathcal{R}}{\leftarrow}} \psi(\psi^{-1}(u_2)) = u_2$ by Proposition 6.3(a). □

**Theorem 6.7** *Confluence (termination, left-most termination, right-most termination) is an undecidable property for length-two SRSs.*

**Proof.** Directly obtained from Theorem 4.4 and Lemma 6.6 (Lemma 6.5). □

## 7 Conclusions

In this paper, we showed that confluence, termination, left-most termination, and right-most termination are undecidable properties for length-two SRSs which are length-preserving SRSs consisting of length-two rules. Thus all these properties are also undecidable properties for linear TRSs in which depth-two variables are allowed in both sides of the rules. We still have remaining questions concerning (un)decidability results for classes of SRSs obtained by limiting the number of rules.

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